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MATHEMATICS

Mathematical Models of Economic Dynamics in the Context of High Inflation and Unstable Development

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Abstract—A new approach to the design of the demand function based on a Pareto distribution law is proposed, and a self-contained mathematical model that adequately describes short-term economic dynamics in the case of high inflation and unstable development is constructed. Classical models well describe only two extreme cases: normal (creeping) inflation and hyperinflation. The model developed has been successfully applied to the analysis and short-term prediction of the Russian economy in the context of high inflation and unstable development, which are intermediate conditions between the above extreme cases.

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At present, the Russian people have been most disturbed by the growth of inflation and recessive economy, which lead to a decline in the standards of living. As based on the 2014 data, for the first time since the 2009 crisis, inflation has grown to two-digit figures (11.4%), while the economic growth has declined from 4.1% (the mean growth rate over 2010–2012) to 0.6%. In the current year, inflation has reached a peak value of 17.5% per year (March, 2015), and the recession in the first half-year exceeded 3%. Expert predictions concerning the recession depth in 2015 range from 3 to 7%, and inflation by the end of the year is predicted to be at 12–16%.

Below, we propose a short-term mathematical model of economic dynamics in the case of high inflation (from 10 to 30% per year) and unstable development. This model is used to predict the rates of inflation and economic growth (recession) for Russia from 2015 to 2018. It is believed that noticeable innovation impulses capable of generating endogenous factors of growth of the Russian economy should not be expected in the nearest one or two years. Under these conditions, emissions of money become the basic source of financing the government budget deficit. The revenue received by the government from an emission of money is called seigniorage. In turn, the emission covering of the government deficit serves is the basic source of the growth of inflation.

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Since the main factor of the inflation rise is the growth of money supply, a key role is played by an equilibrium condition in the money market that is usually written in the general form as (see [1])

$$\left(\frac{M}{P}\right)^S = \left(\frac{M}{P}\right)^D = L(i, Y), \quad (1)$$

where M is the money basis; P is the price level in the economy; i is the nominal interest rate; Y is the real income (GDP); $L(i, Y)$ is the demand function for real money balances; and the superscripts S and D denote supply and demand, respectively. According to the Fisher identity [1], the nominal interest rate (i) is determined by the real interest rate (r) and expected inflation (π^e):

$$i = r + \pi^e; \quad \pi^e = \frac{\dot{P}^e}{P^e}. \quad (2)$$

The development of a particular model of demand for money in high inflation is usually based on the classical demand function for money proposed by Cagan [2] to describe hyperinflation processes, when the prices rise on average by 50% and more within a month:

$$\left(\frac{M}{P}\right)^{\mathrm{D}} = \exp\left(-\alpha\pi^e\right), \quad \alpha > 0. \quad (3)$$

Here, α is the inflation rate elasticity of money demand. The Cagan demand function (3) shows that the demand for money decreases very quickly with growing inflationary expectations (π^e), which is actually observed in the case of hyperinflation. Another successful idea of Cagan is that expectations can be reasonably corrected by applying the adaptive expectation mechanism [3, p. 158]:

$$\dot{\pi}^e = \beta(\pi - \pi^e), \quad (4)$$

where $\pi = \dot{P}/P$ is the real rate of inflation and β is a parameter characterizing the rate at which the economic subjects reconsider their expectations according to the real devaluation of money; here, $\beta > 0$. It is also assumed that the money supply growth rate is a constant; i.e.,

$$\mu = \frac{\dot{M}}{M} = \theta = \text{const}. \quad (5)$$

Cagan's model (3)–(5) has a simple and elegant analytical solution [3, p. 158]:

$$\pi(t) = \theta + (\pi_0 - \theta)\exp\left(-\frac{\beta t}{1 - \alpha\beta}\right). \quad (6)$$

For a hyperinflation economy, we can assume that $\pi_0 > \theta$. If the agents change their expectations in a rational manner, then $\alpha\beta < 1$ and $\pi \to \theta$ as $t \to +\infty$, which agree with conclusions based on the classical quantitative theory of money [3, p. 159]: in equilibrium, $\pi = \mu = \theta$. If the agents sharply change their expectations, then $\alpha\beta > 1$ and $\pi \to +\infty$ as $t \to +\infty$; i.e., the economy cannot reach an equilibrium state. Due to its simplicity and adequacy, Cagan's model has become the most popular demonstration model; so it can be found in all textbooks on macroeconomics (see, for example, [1; 3, pp. 157–159; 4, p. 194]).

A single factor of demand in Cagan's model (3) is inflationary expectations (π^e). Since $\pi^e \gg r$ in the case of hyperinflation, the latter (r) can be neglected. The model does not contain the output (Y) either; it is assumed that there is no economic growth (recession). Obviously, at high inflation rates ($10\% < \pi < 30\%$) when r and π^e are comparable in value, the real interest rate has to be taken into account in demand function (3). Additionally, the economy undergoes significant changes: recessions or expansions. In addition, Cagan assumed that the money supply growth rate (μ) is a constant, which is unacceptable in practice, since μ is a control parameter requiring a flexible regulation policy of the Central Bank of the Russian Federation aimed at stabilizing inflation. Accordingly, it is not surprising that attempts to apply Cagan's model (3)–(5) to the Russian economy have failed [3, 4].

The following money demand function was proposed in [4, p. 194] as applied to the Russian economy in the context of high inflation:

$$\left(\frac{M}{P}\right)^{\mathrm{D}} = \frac{a}{b + c(\pi^e)^2}, \quad (7)$$

where a, b, and c are constant parameters. This function is reduced much more slowly than the exponential demand function in Cagan's model (3). However, it also fails to satisfactorily describe the dynamics of inflation in an unstable economy.

The shortcomings of Cagan's model were partially eliminated in the Bruno–Fischer model [3, pp. 159–164], which includes GDP dynamics and emission financing of the government budget deficit. The money demand function in this model expresses the specific demand in fractions of GDP (Y):

$$\left(\frac{M}{PY}\right)^{\mathrm{D}} = \exp(-\alpha\pi^e), \quad \alpha > 0. \quad (8)$$

It is assumed that the output (Y) grows at a constant rate, i.e., $q_Y = \dot{Y}/Y = \text{const}$. This is typical of a stable economy adapted to hyperinflation, which is a very rare situation. Additionally, the entire government deficit (d in GDP fractions) is assumed to be financed by emission of money:

$$\frac{\dot{M}}{PY} = d = \text{const}. \quad (9)$$

The expectations in the Bruno–Fischer model are adaptive, as in Cagan's model (4). The Bruno–Fischer model (8), (9) no longer gives a simple explicit solution and has to be analyzed using numerical methods. The model well describes hyperinflation, but like Cagan's model, fails in the case of high inflation and unstable development.

If the expected inflation rate is treated as a random variable, which is the case in practice, then we see that the above models involve exponential probability densities (3) and (8) and the Cauchy distribution law (7). All of them have relatively fast decreasing tails, which is confirmed in the case of hyperinflation. This means that the tails of the distributions can be neglected. However, in the case of high inflation, the economy is unstable and there is a high probability of a spike in the expected inflation rate, which has to be taken into account in practical computations. Therefore, distributions with thick tails need to be used in such cases. It is this idea that was first proposed in [5] and was approved by notable economists and mathematicians. To describe the demand function, we propose using a power law:

$$\left(\frac{M}{PY}\right)^{\mathrm{D}} = k(r + \pi^e)^{-\alpha}, \quad \alpha > 0, \quad k = \text{const}, \quad (10)$$

i.e., a Pareto-type distribution function [6, p. 7].

In addition to demand function (10), the key premises of the Cagan and Bruno–Fischer models are accepted. More specifically, following Cagan, we use the adaptive expectation mechanism (4). Following Bruno and Fischer, we assume that the entire government deficit is financed by money emission (9). However, d is not restricted to a constant; instead, we assume that the government will tend to gradually decrease the deficit until a balanced budget is reached in the medium term. Additionally, we consider actual dynamics of GDP ($q_Y = \dot{Y}/Y \neq \text{const}$) and actual strategies for variations in r and $\mu = \dot{M}/M \neq \text{const}$.

Now we will search for the solution of the model. Taking the logarithm of both sides of Eq. (10) yields

$$\ln M - \ln P - \ln Y = \psi - \alpha\ln(r + \pi^e), \quad (11)$$

where $\psi = \ln k$ and $k = e^{\psi}$. Differentiating both sides of Eq. (11), we obtain

$$\mu - \pi - q_Y = -\alpha \frac{\dot{r} + \dot{\pi}^e}{r + \pi^e}. \quad (12)$$

Since we can set $\pi^e = \pi$ in retrospective analysis, Eqs. (11) and (12) can be used to estimate the parameters k and α. Combining Eqs. (9) and (10), we transform the former into

$$\frac{\dot{M}}{PY} = \frac{\dot{M}}{M} \cdot \frac{M}{PY} = \mu k (r + \pi^e)^{-\alpha} = d. \quad (13)$$

Taking the logarithmic derivative of both sides of this equation gives

$$-\alpha \frac{\dot{r} + \dot{\pi}^e}{r + \pi^e} = \frac{\dot{d}}{d} - \frac{\dot{\mu}}{\mu}. \quad (14)$$

Combining Eqs. (12) and (14), we obtain the important equation

$$\pi + q_Y = \mu + \frac{\dot{\mu}}{\mu} - \frac{\dot{d}}{d}, \quad (15)$$

which shows that an unstable non-innovative economy in the context of high inflation is completely determined by two factors: the money supply growth rate (μ) and the government deficit (d).

To separate the rate of inflation (π) and the economic growth (recession) rate (q_Y), which are two variables of interest, we need to derive one more equation. For this purpose, we can use of the Lucas supply equation [1], which describes the deviations of $\bar{Y}$ caused by unexpected deviations of prices (P) in the absence of supply shocks:

$$\ln Y - \ln \bar{Y} = b(\ln P - \ln P^e). \quad (16)$$

Differentiating both sides of this equation yields

$$q_Y = q_{\bar{Y}} + b(\pi - \pi^e). \quad (17)$$

Combining this equation with (4), we obtain

$$q_Y = q_{\bar{Y}} + \rho \dot{\pi}^e, \quad \rho = \frac{b}{\beta}. \quad (18)$$

Now consider Eq. (13), which implies

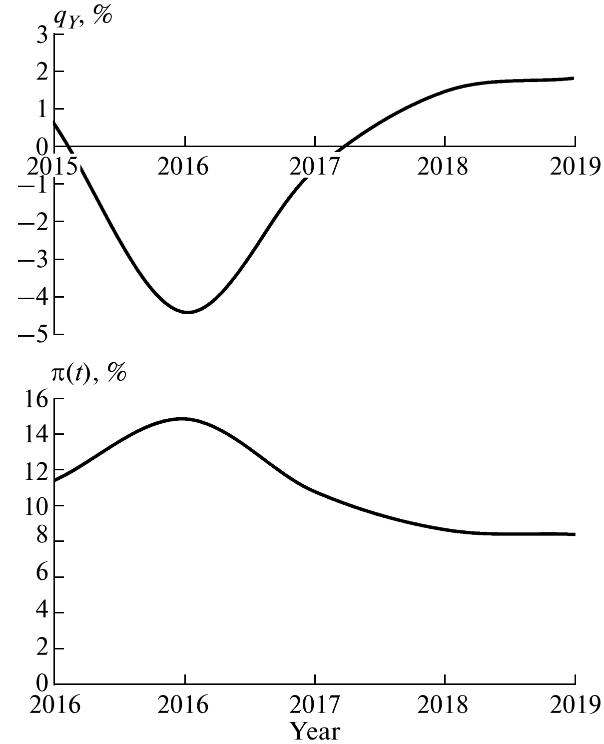
$$\pi^e = \left(\frac{k\mu}{d}\right)^{1/\alpha} - r. \quad (19)$$

Therefore,

$$\dot{\pi}^e = \frac{1}{\alpha}\left(\frac{k\mu}{d}\right)^{\frac{1}{\alpha}} \left(\frac{\dot{\mu}}{\mu} - \frac{\dot{d}}{d}\right) - \dot{r}. \quad (20)$$

Substituting (20) into (18), we obtain the following final equation for computing the dynamics of the economic growth (recession) rate:

$$q_Y = q_{\bar{Y}} + \rho\left[\frac{1}{\alpha}\left(\frac{k\mu}{d}\right)^{\frac{1}{\alpha}} \left(\frac{\dot{\mu}}{\mu} - \frac{\dot{d}}{d}\right) - \dot{r}\right]. \quad (21)$$



Predicted dynamics of the growth rate (q_Y) and the inflation rate (π) for the Russian economy in 2015–2019.

Now, both variables of interest can be determined from this equation, together with (15). Indeed, after calculating the predicted dynamics of q_Y by applying formula (21), the predicted inflation rate (π) can easily be found using formula (15), into which the resulting values of q_Y are substituted.

For prognostic computations based on formula (21), we need to specify scenarios for three key variables: r, μ, and d. A scenario for decreasing the real interest rate, after its sharp rise up to 17% in late 2014, has been formulated by the Central Bank of the Russian Federation, namely, a smooth decrease to a 6% equilibrium level by the end of 2017. We describe this scenario by the logistic function

$$\tilde{r}(t) = \frac{A}{1 + fe^{-ht}} + \Delta, \quad A = 3.87; \quad f = 33.24; \quad h = 1.24; \quad \Delta = 0.055 \quad (22)$$

in 2010–2014 and by a second-degree polynomial

$$\tilde{r}(t) = a_0 + a_1 t + a_2 t^2 \quad (a_0 = 0.17;\ a_1 = -0.0185;\ a_2 = 0.0012) \quad (23)$$

in the prognostic period 2015–2018. For the money supply growth rate, we set a linear growth scenario:

$$\mu = \mu_0 + \mu_1(t - T_0), \quad (24)$$

where μ_0 is the money supply growth rate by early 2015; i.e., T_0 = 2015. Here, $\mu_0 = 0.018$ and $\mu_1 = 0.00013$. In the numerical simulation, the government deficit was set to a constant value of 2.6%, which is the assumed mean over 2014–2018.

Figure 1 shows the predicted dynamics of the rates of inflation and economic growth (recession). It can be seen that the depth of recession will amount to –4.4%, which is expected at the end of 2015. Then the curve begins to increase and the recession is replaced by growth reaching a rate of about 1.5% in 2017 and 1.8% in 2018. However, a 0.7% recession will still be observed in 2016. The inflation rate will be 14.8% by the end of 2015 and will continue to reduce down to 10.8% by the end 2016 and to 8% in early 2018. The key parameters α and k in (10) were estimated using Eqs. (11) and (12): $\alpha = 0,33$ and $k = 0,25$. Then the potential equilibrium rate of growth $q_{\bar{Y}}$ of the Russian economy and the parameter ρ were estimated with the help of Eq. (18) with actual data for $\pi = \pi^e$ used in the retrospective period: $q_{\bar{Y}} = 0.98\%$ in 2014–2015 and $\rho = -1.4$. Note that a detailed study was performed in [7, p. 27], where it was shown that the potential (structural) growth rate of the Russian economy steadily reduced from 4.3% in 2009 to 1–2% in 2014. It can be seen that our estimate $q_{\bar{Y}} \cong 1\%$ for 2014–2015 agrees with the results of [7].

CONCLUSIONS

1. In the case of a non-innovative economy, it follows from Eq. (15) that the rates of inflation and economic growth (recession) are determined only by the money supply growth rate and the government deficit. Therefore, the Government of the Russian Federation should take all measures to reduce the government deficit in order to reduce inflation and revive economic growth.

2. In the case of high inflation and government deficit, the economic growth equation (21) implies that an increase in the money supply growth rate in the short term facilitates a rise in the economic growth rate, which agrees with Nobel winner Lucas' conclusions.

3. The economic growth equation (21) also implies that a decrease in the money supply growth rate in the context of high inflation and instability leads to a noticeable decline in the economic growth rate. This attests to a liquidity deficit in the Russian economy and to its low monetization.

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